HARMONIC MAPS ON LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

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Abstract. We investigate harmonic maps on almost contact metric manifolds which are locally conformal to almost cosymplectic manifolds. We obtain the necessary and sufficient conditions for the holomorphy to imply harmonicity and then we find obstructions to the existence of non-constant pluriharmonic maps. We also establish some results on the stability of the identity map on a locally conformal almost cosymplectic manifold of pointwise constant ϕ-holomorphic sectional curvature.

Keywords: Harmonic map; locally conformal almost cosymplectic manifold; φ-holomorphic sectional curvature; holomorphic map; stability.

Mathematics Subject Classification (MSC2020): 53C55; 53C43; 58E20.

1. Introduction

If (ϕ, ξ, η, g) is an almost contact metric structure on a smooth manifold M, then a conformal change of the metric g leads to a new metric which is no longer compatible with the almost contact structure (ϕ, ξ, η). As shown in [33], this incompatibility can be corrected by conveniently changing ξ and η. This motivated the introduction of conformal changes of an almost contact metric structure by Vaisman [33] and subsequently the study of such conformal changes become a fervent topic. In particular, the question of elucidating geometry of almost contact metric manifolds which are locally conformal to almost cosymplectic manifolds has been investigated by various authors (see, e.g., [6, 7, 11, 20, 22, 23, 27, 28]).

On the other hand, the study of harmonic maps between Riemannian manifolds is a topic of high interest due to their applications in various fields of science, including theoretical physics, computational physics, chemistry, fluid mechanics and computer graphics. As pointed out in [15], covering all these aspects is difficult to achieve even in an entire book. There is an abundant literature regarding the harmonic maps between manifolds equipped with almost contact structures and compatible metrics, but the case of locally conformal almost cosymplectic manifolds was not tackled mainly due to the difficulty in using the tensorial equations that governs the geometry of such spaces in applying any criterion for the study of harmonicity. The aim of the present work is to fill this gap.

In the first part of the article, we derive the conditions under which the holomorphy of a map from a locally conformal almost cosymplectic manifold onto a cosymplectic manifold implies the harmonicity. Then we investigate a natural problem, namely under what conditions a harmonic map becomes pluriharmonic. In particular, we obtain obstructions to the existence of non-constant pluriharmonic maps. In the last part of the article, we find conditions under which the identity map of a normal locally conformal cosymplectic manifold of pointwise constant ϕ-holomorphic sectional curvature is stable, respectively unstable.

2. Preliminaries

2.1. **Manifolds equipped with almost contact metric structures.** Suppose M is an almost contact metric manifold of dimension $(2n+1)$ equipped with the almost contact metric structure (ϕ, ξ, η, g). Then it is known that ϕ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form on M, satisfying [2]

$$\begin{aligned} &\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = \eta \circ \phi = 0 \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad (1) \\ &\eta(X) = g(X, \xi) \end{aligned}$$

for any vector fields X, Y on M, where g denotes the Riemannian metric on M and I stands for the identity endomorphism on the tangent bundle TM. It is clear that the tangent bundle of any almost contact metric manifold splits as the orthogonal sum $TM = \mathcal{D} \oplus \mathrm{Span}\{\xi\}$, where $\mathcal{D} = \mathrm{Ker}\eta$ is called the *contact distribution*.

Let us denote by Φ the fundamental 2-form of the almost contact metric manifold M. This 2-form is given by

$$\Phi(X, Y) = g(\varphi X, Y),$$

for any vector fields X, Y on M. Then $M(\phi, \xi, \eta, g)$ is said to be an *almost cosymplectic manifold* if η and Φ are both closed forms. If an almost cosymplectic manifold is normal, i.e. $[\phi, \phi] + 2\xi \otimes d\eta = 0$ (where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y],$$

for any vector fields X, Y on M), then $M(\phi, \xi, \eta, g)$ is said to be a *cosymplectic manifold*. It is known that a necessary and sufficient condition for an almost contact metric manifold to be cosymplectic is that the Levi-Civita connection ∇ of the metric g satisfy [2]

$$\nabla\phi = 0. \tag{2}$$

It is known that any cosymplectic manifold is locally a product manifold of a Kähler manifold and $\mathbb{R}$. Moreover, we note that the product manifold of an almost Kähler manifold and $\mathbb{R}$ provides the simplest example of almost cosymplectic manifold, but there exist almost cosymplectic manifolds which are not such product manifolds (for various examples see [3, 25, 26]).

An almost contact metric manifold M endowed with the almost contact metric structure (ϕ, ξ, η, g) is said to be a *locally conformal (almost) cosymplectic manifold* if there exists an open covering $\{U_\alpha\}$ of M equipped with differentiable functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that the almost contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g_\alpha)$ defined on each U_α by

$$\phi_\alpha = \phi,\ \xi_\alpha = e^{\sigma_\alpha}\xi,\ \eta_\alpha = e^{-\sigma_\alpha}\eta,\ g_\alpha = e^{-2\sigma_\alpha}g,$$

is (almost) cosymplectic (see [27]). We recall that the above concept was originally introduced and investigated by Vaisman [33], who obtained the following characterization of locally conformal almost cosymplectic manifolds.

Theorem 2.1. [33] *An almost contact metric manifold M with the almost contact metric structure (ϕ, ξ, η, g) is locally conformal almost cosymplectic manifold if and only if there exists a 1-form ω on M such that*

$$d\Phi = 2\omega \wedge \Phi,\ d\eta = \omega \wedge \eta,\ d\omega = 0. \tag{3}$$

Note that if the 1-form ω satisfying (3) exists, then this 1-form is unique. Hence, ω is a characteristic form of a locally conformal cosymplectic manifold, known as the *Lee form* of the locally conformal almost cosymplectic manifold $M(\phi, \xi, \eta, g)$. Locally, ω is given by $\omega|_{U_\alpha} = d\sigma_\alpha$ (see [28]). Several examples of locally conformal almost cosymplectic manifolds can be found in [7, 22, 27, 28]. At this point, we would like to recall only a particular class of locally conformal cosymplectic manifolds which are foliated by generalized Hopf manifolds [21]. For such a manifold, which is called a *PC-manifold*, it was proved by Marrero that the universal covering space is the product of a c-Sasakian manifold with the 2-dimensional hyperbolic space. Recall that an almost contact metric manifold $M(\phi, \xi, \eta, g)$ is called c-Sasakian, where c is a nonzero real number, if M is normal and $d\eta = c\Phi$. Clearly, a 1-Sasakian manifold is nothing but a Sasakian manifold (for examples of Sasakian manifolds and Sasakian space forms see [2]). Note that examples of c-Sasakian manifolds of constant ϕ-sectional curvature can be constructed in a standard manner for any $c \neq 0$. Actually, if $M(\phi, \xi, \eta, g)$ is a Sasakian manifold of constant ϕ-sectional curvature k, then $M(\phi', \xi', \eta', g')$ is a c-Sasakian manifold of constant ϕ-sectional curvature kc^2, usually denoted by $M(c, kc^2)$, where

$$\phi' = \phi,\ \xi' = c\xi,\ \eta' = \frac{1}{c}\eta,\ g' = \frac{1}{c^2}g.$$

Next we recall two results obtained by Olszak [27] that will help us later.

Theorem 2.2. [27] *An almost contact metric manifold M with the almost contact metric structure (ϕ, ξ, η, g) is locally conformal cosymplectic manifold if and only if there exists a 1-form ω on M such that $d\omega = 0$ and*

$$(\nabla_X \phi)Y = \omega(\phi Y)X - \omega(Y)\phi X - g(X, \phi Y)B + g(X, Y)\phi B, \tag{4}$$

for any vector fields X, Y on M, where $B = \omega^\#$ (here $^\#$ signifies the rising of the indices with respect to the metric g).

Note that $B = \omega^\#$ is called the *Lee vector field* of the locally conformal cosymplectic manifold $M(\phi, \xi, \eta, g)$, by analogy with the locally conformal Kähler case [9].

Theorem 2.3. [27] *Let $M(\phi, \xi, \eta, g)$ be an almost contact metric manifold. Then the next assertions are equivalent.*

(1) *$M(\phi, \xi, \eta, g)$ is a normal locally conformal cosymplectic manifold.*
(2) *$M(\phi, \xi, \eta, g)$ is a locally conformal cosymplectic manifold with $\omega = f\eta$.*
(3) *There exists a function f satisfying $df \wedge \eta = 0$ such that*

$$(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{5}$$

for any vector fields X, Y on M.

We remark that the family of normal locally conformal cosymplectic manifolds includes the class of α-Kenmotsu manifolds (for details on the last mentioned class of manifolds see [17]). In fact, an α-Kenmotsu manifold is an almost contact metric manifold characterized by the equation (5) in the above theorem, with $f = \alpha$, where α is a non-zero constant. Note that a 1-Kenmotsu manifold is nothing but a Kenmotsu manifold [19]. Motivated by the above facts and Theorem 2.3, a normal locally conformal cosymplectic manifold is also called an f-Kenmotsu manifold. Nontrivial examples of such manifolds can be find in [28].

2.2. **Harmonic maps and stability.** The second fundamental form α_F of a smooth map $F:(M,g)\to(N,h)$ between two Riemannian manifolds (M,g) and (N,h) is given by:

$$\alpha_F(X,Y)=\widetilde{\nabla}_X F_*Y-F_*\nabla_X Y, \tag{6}$$

for any vector fields X,Y on M, where ∇ denotes the Riemannian connection on M and $\widetilde{\nabla}$ is the pullback of the Riemannian connection ∇' on N to the induced vector bundle $F^{-1}(TN)$, that is,

$$\widetilde{\nabla}_X F_*Y=\nabla'_{F_*X}F_*Y. \tag{7}$$

The tension field $\tau(F)$ of the map F is defined as the trace of the second fundamental form α_F:

$$\tau(F)_p=\sum_{i=1}^{m}\alpha_F(e_i,e_i), \tag{8}$$

where $\{e_1,e_2,...,e_m\}$ is a local orthonormal frame of T_pM, $p\in M$.

If the Riemannian manifold (M,g) is compact, then the energy of the map $F:(M,g)\to(N,h)$ is defined as

$$\mathcal{E}(F)=\int_M e(F)\vartheta_g,$$

where ϑ_g denotes the canonical measure associated with the Riemannian metric g and

$$e(F)_p=\frac{1}{2}Trace_g(F^*h)_p,\ \forall p\in M.$$

Recall that the smooth map $F:(M,g)\to(N,h)$ is called *harmonic* if F is a critical point of $\mathcal{E}$. The following criterion for the study of harmonicity is extremely useful: $F:(M,g)\to(N,h)$ is a harmonic map if and only if $\tau(F)$ vanishes at each point $p\in M$ (see [10]). The identity map on a compact Riemannian space is one of the simplest example of harmonic maps. For other examples, see [12, 18, 31, 32, 34]

Let us consider now a smooth two-parameter variation $\{F_{s,t}\}_{s,t\in(-\epsilon,\epsilon)}$ of a harmonic map F such that $F_{0,0}=F$ and let $V,W\in\Gamma(f^{-1}(TN))$ be the corresponding variational vector fields, i.e. $V=\frac{\partial}{\partial s}(F_{s,t})|_{(s,t)=(0,0)}$, $W=\frac{\partial}{\partial t}(F_{s,t})|_{(s,t)=(0,0)}$. Recall that the *Hessian* of F is given by:

$$Hess_F(V,W)=\frac{\partial^2}{\partial s\partial t}(\mathcal{E}(F_{s,t}))|_{(s,t)=(0,0)},$$

while the *index* $\mathcal{I}_F$ of F is defined as the dimension of the largest subspace of $\Gamma(F^{-1}(TN))$ on which $Hess_F$ is negative definite. A harmonic map F is called *stable* if $\mathcal{I}_F = 0$; otherwise, F is called *unstable*. The following formula, known as the *second variation formula*, was obtained by Mazet and Smith (see [24, 30]):

$$Hess_F(V,W) = \int_M h(\mathcal{J}_F(V), W)\vartheta_g, \tag{9}$$

where $\mathcal{J}_F$ stands for the Jacobi operator of the map F (see also [1]). Recall that $\mathcal{J}_F$ is a differential operator on the space of variation vector fields along F, defined as

$$\mathcal{J}_F(V) = -\sum_{i=1}^{m}(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i}e_i})V - \sum_{i=1}^{m} R^N(V, F_*e_i)F_*e_i,$$

for any section V of $F^{-1}(TN)$, where R^N is the curvature tensor of (N,h) and $\{e_1,\ldots,e_m\}$ is a local orthonormal frame on M. Note that the operator given by

$$\bar{\Delta}_F V = -\sum_{i=1}^{m}(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i}e_i})V$$

is known as the rough Laplacian. Therefore, it is obvious that the Jacobi operator of the map F can be expressed as

$$\mathcal{J}_F(V) = \bar{\Delta}_F V - \sum_{i=1}^{m} R^N(V, F_*e_i)F_*e_i.$$

3. Harmonicity and pluriharmoncity of holomorphic maps from locally conformal almost cosymplectic manifolds

Let $F : M \to N$ be a smooth map between two almost contact metric manifolds $M(\phi_1,\xi_1,\eta_1,g)$ and $N(\phi_2,\xi_2,\eta_2,h)$. Then F is said to be a (ϕ_1,ϕ_2)-holomorphic map if $F_*\circ\phi_1 = \phi_2\circ\circ F_*$ (cf. [5]). Particular and important examples of such maps are provided by the almost contact metric submersions [8, 12]. In particular, several examples of such submersions were constructed in [4, Section 5].

Another example of (ϕ_1,ϕ_2)-holomorphic map can be obtained as follows. Let M be a manifold equipped with an almost contact structure (ϕ_1,ξ_1,η_1,g) and let N be an invariant submanifold of M (that is, $\phi_1(T_pN)\subset T_pN$, for any $p\in N$), tangent to ξ_1. Suppose $i : M \to N$ is the inclusion map. Then taking the restriction of (ϕ_1,ξ_1,η_1,g) to N we obtain an almost contact structure (ϕ_2,ξ_2,η_2,h) on N, and i is a (ϕ_1,ϕ_2)-holomorphic map.

Note that several examples of (ϕ_1, ϕ_2)-holomorphic maps that are not Riemannian submersions can be found in [8, Section 3]. In particular, we have that the map $\pi : S^{2n+1}(c,k) \times \mathbb{H}^2_c \to \mathbb{C}P^n(k+3c^2) \times \mathbb{R}$, defined by

$$\pi(x,(u,v)) = (\bar{\pi}(x), v),\ x \in S^{2n+1}(c,k),\ (u,v) \in \mathbb{H}^2_c,$$

where $\bar{\pi}$ is the Hopf fibration of the c-Sasakian manifold $S^{2n+1}(c,k)$ onto the n-dimensional complex projective space $\mathbb{C}P^n(k+3c^2)$ of positive holomorphic sectional curvature $k+3c^2$ and $\mathbb{H}^2_c$ is the 2-dimensional hyperbolic space (i.e. the space of 2-tuples of real numbers (u,v) equipped with the Riemannian metric $du^2 + e^{-2cu}dv^2$, where c is a positive constant), is a (ϕ_1, ϕ_2)-holomorphic map.

Proposition 3.1. *Let $F : M \to N$ be a (ϕ_1, ϕ_2)-holomorphic map from a locally conformal almost cosymplectic manifold $M(\phi_1, \xi_1, \eta_1, g)$ to a cosymplectic manifold $N(\phi_2, \xi_2, \eta_2, h)$. Then the map F is harmonic if and only if $B = \omega^\#$ belongs to the kernel of F_*, where ω is the characteristic vector field of M.*

Proof. Using (6) and (7) we obtain

$$\begin{aligned} \phi_2(\alpha_F(X,Y)) + (\widetilde{\nabla}_X\phi_2)(F_*Y) &= \phi_2(\widetilde{\nabla}_X F_*Y - F_*\nabla_X Y) \\ &\quad + \widetilde{\nabla}_X \phi_2 F_*Y - \phi_2 \widetilde{\nabla}_X F_*Y \\ &= -\phi_2 F_* \nabla_X Y + \nabla'_{F_*X}\phi_2 F_*Y, \end{aligned}$$

for any vector fields X, Y and M.

Taking now into account that F is a (ϕ_1, ϕ_2)-holomorphic, the above equation implies

$$\phi_2(\alpha_F(X,Y)) + (\widetilde{\nabla}_X\phi_2)(F_*Y) = \nabla'_{F_*X} F_*\phi_1 Y - F_*\phi_1 \nabla_X Y. \tag{10}$$

On the other hand, using again (6), (7) and the (ϕ_1, ϕ_2)-holomorphicity property of F, it is easy to check that

$$F_*\left((\nabla_X \phi_1)Y\right) + \alpha_F(X, \phi_1 Y) = \nabla'_{F_*X} F_*\phi_1 Y - F_*\phi_1 \nabla_X Y. \tag{11}$$

From (10) and (11), we deduce

$$\phi_2(\alpha_F(X,Y)) + (\widetilde{\nabla}_X\phi_2)(F_*Y) = F_*\left((\nabla_X \phi_1)Y\right) + \alpha_F(X, \phi_1 Y). \tag{12}$$

Now, we consider an adapted orthonormal frame on M:

$$\{E_1, \ldots, E_n, E_{n+1} = \phi_1 E_1, \ldots, E_{2n} = \phi_1 E_n, E_{2n+1} = \xi_1\}.$$

Replacing $X = Y = E_i$ in (12) and using (7) and the fact that $\nabla'\phi_2 = 0$ (as $N(\phi_2, \xi_2, \eta_2, h)$ is a cosymplectic manifold), we derive after summing up over $i = 1, \dots, 2n+1$ that

$$\phi_2(\tau_F) = F_*(\mathrm{div}\phi_1) + \sum_{i=1}^{2n+1} \alpha_F(E_i, \phi_1 E_i). \tag{13}$$

But the last sum in (13) can be computed as follows, taking into account that $E_{n+i} = \phi_1 E_i$ $(i = 1, \dots, n)$, $E_{2n+1} = \xi_1$ and using (1):

$$\begin{aligned}\sum_{i=1}^{2n+1} \alpha_F(E_i, \phi_1 E_i) &= \sum_{i=1}^{n} \alpha_F(E_i, \phi_1 E_i) + \sum_{i=1}^{n} \alpha_F(\phi_1 E_i, \phi_1^2 E_i) + \alpha_F(\xi_1, \phi_1 \xi_1) \\ &= \sum_{i=1}^{n} \alpha_F(\phi_1 E_i, \eta_1(E_i)\xi_1).\end{aligned}$$

It is now easy to see that, in view of (1), we have

$$\eta_1(E_i) = g(E_i, \xi_1) = 0,\ i = 1, \dots, n$$

and therefore (13) reduces to

$$\phi_2(\tau_F) = F_*(\mathrm{div}\phi_1). \tag{14}$$

Using now the fact that $M(\phi_1, \xi_1, \eta_1, g)$ is a locally conformal cosymplectic manifold, we can state easily from (4) that

$$(\nabla_{E_i}\phi_1)E_i = \omega(E_{n+i})E_i - \omega(E_i)E_{n+i} + \phi_1 B,\ i = 1, \dots, n, \tag{15}$$

$$(\nabla_{E_{n+i}}\phi_1)E_{n+i} = -\omega(E_i)E_{n+i} + \omega(E_{n+i})E_i + \phi_1 B,\ i = 1, \dots, n \tag{16}$$

and

$$(\nabla_{E_{2n+1}}\phi_1)E_{2n+1} = \phi_1 B. \tag{17}$$

Using (15), (16) and (17), we get

$$\begin{aligned}\mathrm{div}\phi_1 &= \sum_{i=1}^{2n+1} (\nabla_{E_i}\phi_1)E_i \\ &= \sum_{i=1}^{n} \left[(\nabla_{E_i}\phi_1)E_i + (\nabla_{E_{n+i}}\phi_1)E_{n+i}\right] + (\nabla_{\xi_1}\phi_1)\xi_1 \\ &= (2n+1)\phi_1 B + 2\sum_{i=1}^{n} [\omega(E_{n+i})E_i - \omega(E_i)E_{n+i}]\end{aligned}$$

and since $B = \omega^{\#}$, in view of (1) we derive

$$\begin{aligned} \mathrm{div}\phi_1 &= (2n+1)\phi_1 B + 2\sum_{i=1}^{n}[g(B, E_{n+i})E_i - g(B, E_i)E_{n+i}] \\ &= (2n+1)\phi_1 B + 2\sum_{i=1}^{n}[g(B, \phi_1 E_i)E_i - g(B, E_i)\phi_1 E_i] \\ &= (2n+1)\phi_1 B - 2\sum_{i=1}^{n}[g(\phi_1 B, E_i)E_i + g(\phi_1 B, \phi_1 E_i)\phi_1 E_i] \\ &= (2n+1)\phi_1 B - 2\phi_1 B \\ &= (2n-1)\phi_1 B. \end{aligned}$$

Finally, from the last equation and (14), we obtain that

$$\phi_2(\tau_F) = (2n-1)F_*(\phi_1 B)$$

and using again the (ϕ_1, ϕ_2)-holomorphicity property of F, we derive

$$\phi_2(\tau_F) = (2n-1)\phi_2 F_* B. \tag{18}$$

Consequently, we have form (18) that the map F is harmonic if and only if $F_* B = 0$ and as $B = \omega^{\#}$ we arrive at the desired conclusion. □

Remark 3.2. Proposition 3.1 is the counterpart of [13, Proposition 4.1] in the contact geometry.

Remark 3.3. Proposition 3.1 provides us a criterion to decide when a (ϕ_1, ϕ_2)-holomorphic map is harmonic. Applying this criterion, we derive that the locally conformal cosymplectic submersions investigated in [8, Section 2] cannot be harmonic, due to the fact that for such submersions $B = \omega^{\#}$ is an horizontal vector field.

A smooth map $F : M \to N$ between two almost contact metric manifolds $M(\phi_1, \xi_1, \eta_1, g)$ and $N(\phi_2, \xi_2, \eta_2, h)$ is called Φ-*pluriharmonic* if the second fundamental form α_F of F satisfies

$$\alpha_F(X, Y) + \alpha_F(\phi_1 X, \phi_1 Y) = 0, \tag{19}$$

for any vector fields X, Y on M (see [16]). Furthermore, F is called $\mathcal{D}_1$-*pluriharmonic* if (19) is valid only for $X, Y \in \Gamma(\mathcal{D}_\infty)$, where $\mathcal{D}_1$ is the contact distribution on M. Obviously, any Φ-pluriharmonic map is $\mathcal{D}_1$-pluriharmonic.

Remark 3.4. It is easy to check that any ϕ_1-*pluriharmonic* map $F : M \to N$ between two almost contact metric manifolds $M(\phi_1, \xi_1, \eta_1, g)$ and $N(\phi_2, \xi_2, \eta_2, h)$ is harmonic. In order to prove this, let us consider $\{E_1, \ldots, E_n, E_{n+1} = \phi_1 E_1, \ldots, E_{2n} = \phi_1 E_n, E_{2n+1} = \xi_1\}$ be an adapted orthonormal frame on M. Then by replacing $X = Y = E_i$ in (19) for $i = 1, \ldots, n$, we get

$$\alpha_F(E_i, E_i) + \alpha_F(\phi_1 E_i, \phi_1 E_i) = 0. \tag{20}$$

On the other hand, if we replace $X = Y = E_{2n+1}$ in (19), in view of (1) we obtain

$$\alpha_F(\xi_1, \xi_1) = 0. \tag{21}$$

Therefore, using (8), (20) and (21), we derive immediately that $\tau(F)$ vanishes on M. Consequently, F is harmonic. Now, it is natural to investigate under what conditions a harmonic map is ϕ_1-pluriharmonic. At this point, we are able to prove the next result concerning ϕ_1-pluriharmonicity of (ϕ_1, ϕ_2)-holomorphic maps.

Proposition 3.5. *Assume $F : M \to N$ is a (ϕ_1, ϕ_2)-holomorphic map from a locally conformal cosymplectic manifold $M(\phi_1, \xi_1, \eta_1, g)$ to a cosymplectic manifold $N(\phi_2, \xi_2, \eta_2, h)$ such that B belongs to the contact distribution $\mathcal{D}_1 = \mathrm{Ker}\eta_1$. If F is a ϕ_1-pluriharmonic map, then $F_*|_{\mathrm{Ker}\eta_1}$=0.*

Proof. From (12), using (7) and taking into account that M is locally conformal cosymplectic and N is cosymplectic, we obtain in view of Theorem 2.2:

$$\begin{aligned} \phi_2(\alpha_F(X,Y)) \;&=\; F_*\left(\omega(\phi_1 Y)X - \omega(Y)\phi_1 X\right) \\ &\quad + F_*\left(-g(X,\phi_1 Y)B + g(X,Y)\phi_1 B\right) \\ &\quad + \alpha_F(X, \phi_1 Y), \end{aligned} \tag{22}$$

for any vector fields X, Y on M. Replacing now (X, Y) in (22) by $(\phi_1 X, \phi_1 Y)$, where $X, Y \in \mathcal{D}_1 = \mathrm{Ker}\eta_1$, and using (1), we get

$$\begin{aligned} \phi_2(\alpha_F(\phi_1 X, \phi_1 Y)) \;&=\; F_*\left(-\omega(Y)\phi_1 X + \omega(\phi_1 Y)X\right) \\ &\quad + F_*\left(g(\phi_1 X, Y)B + g(X,Y)\phi_1 B\right) \\ &\quad - \alpha_F(\phi_1 X, Y). \end{aligned} \tag{23}$$

Due to the symmetry of α_F, by adding (22) and (23) we derive

$$\begin{aligned} \phi_2(\alpha_F(X,Y) + \alpha_F(\phi_1 X, \phi_1 Y)) \;&=\; F_*\left(2g(X,Y)\phi_1 B\right) \\ &\quad - F_*\left(\omega(Y)\phi_1 X + \omega(X)\phi_1 Y\right) \\ &\quad + F_*\left(\omega(\phi_1 Y)X + \omega(\phi_1 X)Y\right). \end{aligned} \tag{24}$$

Now, as F is ϕ_1-pluriharmonic, it follows that F is harmonic and in view of Proposition 3.1, we have that $B = \omega^\# \in \mathrm{Ker}F_*$. Therefore, due to the (ϕ_1, ϕ_2)-holomorphicity of F, we obtain form (24) that

$$\alpha_F(X,Y) + \alpha_F(\phi_1 X, \phi_1 Y) = -F_*\left(\omega(Y)X + \omega(X)Y\right) - F_*\left(\omega(\phi_1 Y)\phi_1 X + \omega(\phi_1 X)\phi_1 Y\right). \tag{25}$$

Consequently, as F is ϕ_1-pluriharmonic (and in particular harmonic), we obtain from (25) taking $Y = B$ and $X \in \mathcal{D}_1 = \mathrm{Ker}\eta_1$, X orthogonal to B, that $F_*X = 0$. But, as we have $B \in \mathcal{D}_1$ and $F_*B = 0$, we conclude that $F_*|_{\mathrm{Ker}\eta_1}=0$. □

An immediate consequence of Proposition 3.5 is the following result concerning the constancy of ϕ_1-pluriharmonic maps.

Corollary 3.6. *Let $F : M \to N$ be a (ϕ_1, ϕ_2)-holomorphic map from a locally conformal almost cosymplectic manifold $M(\phi_1, \xi_1, \eta_1, g)$ to a cosymplectic manifold $N(\phi_2, \xi_2, \eta_2, h)$ such that B belongs to the contact distribution $\mathcal{D}_1 = \mathrm{Ker}\eta_1$. Then F is $\mathcal{D}_1$-pluriharmonic if and only if $F_*|_{\mathcal{D}_1}=0$. Moreover, if $\xi_1 \in \mathrm{Ker}F_*$, then F is ϕ_1-pluriharmonic if and only if F is constant.*

Remark 3.7. If $M(\phi_1, \xi_1, \eta_1, g)$ is a PC-manifold, then it is known that the Lee form ω satisfies $\omega(\xi_1) = 0$ (see [21]). But it is easy to see that this implies $B = \omega^\#$ belongs to the contact distribution $\mathcal{D}_1 = \mathrm{Ker}\eta_1$. Consequently, if we consider the PC-manifold $M = \mathbb{R}^{2n+1}(c) \times \mathbb{H}^2_c$, we deduce that the map $\pi : M \to \mathbb{C}^m \times \mathbb{R}$, defined by

$$\pi(x,(u,v)) = (\bar{\pi}(x), v),\ x \in \mathbb{R}^{2n+1}(c),\ (u,v) \in \mathbb{H}^2_c,$$

where $\bar{\pi}$ is the canonical projection of the c-Sasakian manifold $\mathbb{R}^{2n+1}(c)$ onto the m-dimensional flat Kählerian space $\mathbb{C}^m \simeq \mathbb{R}^{2m}$ (where $m \leq n$), is a (ϕ_1, ϕ_2)-holomorphic map satisfying the hypotheses of Corollary 3.6. As π is not a constant map, in view of Corollary 3.6, we conclude that π is not ϕ_1-pluriharmonic.

4. Stability results for identity map of compact locally conformal almost cosymplectic manifolds

Suppose $M(\phi, \xi, \eta, g)$ is an almost contact metric manifold. A 2-plane Π of T_pM, where $p \in M$, is said to be a *ϕ-holomorphic* plane if Π is orthogonal to ξ and $\phi(\Pi) \subset \Pi$. As usual, let us denote by $K(\Pi)$ the sectional curvature of Π. The manifold $M(\phi, \xi, \eta, g)$ is said to be of *pointwise constant ϕ-holomorphic sectional curvature* if at any point p of M, we have that $K(\Pi)$ is independent on the choice of Π as holomorphic plane in T_pM. In this case, the function k defined by $k(p) = K(\Pi)$, is said to be the *ϕ-holomorphic sectional curvature* of M.

It was proved in [27] that a normal locally conformal cosymplectic manifold with dimension ≥ 5 has pointwise constant ϕ-holomorphic sectional curvature if and only if its Riemannian curvature tensor is

$$\begin{aligned} R(X,Y,Z,W) \;=\; & \frac{k-3f^2}{4}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)] \\ & +\frac{k+f^2}{4}[g(X,\phi W)g(Y,\phi Z)-g(X,\phi Z)g(Y,\phi W) \\ & -2g(X,\phi Y)g(Z,\phi W)] \\ & -\frac{k+f^2+4f'}{4}[g(X,W)\eta(Y)\eta(Z)-g(X,Z)\eta(Y)\eta(W) \\ & +g(Y,Z)\eta(X)\eta(W)-g(Y,W)\eta(X)\eta(Z)], \end{aligned} \tag{26}$$

where f is the function satisfying $\omega = f\eta$, $f' = \xi f$ and k is the ϕ-holomorphic sectional curvature of the manifold M (see [27, Theorem 5.3]).

A concrete example of such manifold can be obtained taking the product of a complex space form $N(c)$ of constant holomorphic sectional curvature c and complex dimension ≥ 2, with the real line $\mathbb{R}$. Then it is known that $M = N(c) \times \mathbb{R}$ admits a canonical cosymplectic structure (ϕ,ξ,η,g) and transforming conformally this structure through a function σ that depends only on s (the coordinate on $\mathbb{R}$), then one arrives at a normal locally conformal structure (ϕ',ξ',η',g') of pointwise constant ϕ-holomorphic sectional curvature on M, given by

$$\phi' = \phi,\ \xi' = e^{-\sigma}\xi,\ \eta' = e^{\sigma}\eta, g' = e^{2\sigma}g.$$

Note that if the function σ is periodic, then the above locally conformal cosymplectic structure can be projected on $M' = N(c) \times S^1$. At this point we would like to emphasize that although there exist examples of normal locally conformal cosymplectic manifolds of pointwise constant ϕ-holomorphic sectional curvature, it is an open problem to establish the existence of non-normal examples [27].

In the following, we are interested in obtaining conditions under which the identity map of a normal locally conformal cosymplectic manifold $M(\phi,\xi,\eta,g)$ of pointwise constant ϕ-holomorphic sectional curvature is stable, respectively unstable.

Theorem 4.1. *Let $M(\phi,\xi,\eta,g)$ be a compact normal locally conformal cosymplectic manifold of dimension $2n+1 \geq 5$ and of pointwise constant ϕ-holomorphic sectional curvature k. Suppose the following conditions are satisfied:*

(i) $(n+1)k + (1-3n)f^2 - 2f' \geq 0$*;*

(ii) $(n+1)(k+f^2)+2(2n-1)f' \leq 0$.

Then the identity map 1_M is weakly stable.

Proof. Let $\{E_1,\dots,E_n,E_{n+1}=\phi E_1,\dots,E_{2n}=\phi E_n,E_{2n+1}=\xi\}$ be an adapted orthonormal frame on M. Then using (26), we obtain for any vector field V on M and $i=1,\dots,n$ that:

$$
\begin{aligned}
g(R(E_i,V)E_i,V) &= \frac{k-3f^2}{4}\left[g(V,V)-g^2(E_i,V)\right] \\
&\quad +\frac{3(k+f^2)}{4}g^2(\phi V,E_i)-\left(\frac{k+f^2}{4}+f'\right)\eta^2(V), \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
g(R(\phi E_i,V)\phi E_i,V) &= \frac{k-3f^2}{4}\left[g(V,V)-g^2(\phi E_i,V)\right] \\
&\quad +\frac{3(k+f^2)}{4}g^2(\phi V,\phi E_i)-\left(\frac{k+f^2}{4}+f'\right)\eta^2(V). \qquad (28)
\end{aligned}
$$

Also, we derive

$$
(29)\; g(R(E_{2n+1},V)E_{2n+1},V) = -\left(f^2+f'\right)\left[g(V,V)-\eta^2(V)\right].
$$

Using now (27), (28) and (29), we obtain

$$
\begin{aligned}
\sum_{i=1}^{2n+1} g(R(E_i,V)E_i,V) &= \left[\frac{(n+1)k-(3n-1)f^2}{2}-f'\right]g(V,V)- \\
&\quad -\left[\frac{(n+1)(k+f^2)}{2}+(2n-1)f'\right]\eta^2(V). \qquad (30)
\end{aligned}
$$

Now, the second variation formula and (30) imply

$$
\begin{aligned}
Hess_{1_M}(V,V) &= \int_M h(\widetilde{\nabla}V,\widetilde{\nabla}V)\vartheta_g \\
&\quad +\int_M \left[\frac{(n+1)k-(3n-1)f^2}{2}-f'\right]g(V,V)\vartheta_g \\
&\quad -\int_M \left[\frac{(n+1)(k+f^2)}{2}+(2n-1)f'\right]\eta^2(V)\vartheta_g. \qquad (31)
\end{aligned}
$$

Therefore, we conclude from (31) that the identity map 1_M is weakly stable, provided that both conditions (i) and (ii) are satisfied. □

Theorem 4.2. *Let $M(\phi,\xi,\eta,g)$ be a compact normal locally conformal cosymplectic manifold of dimension $2n+1\geq 5$ and of pointwise constant ϕ-holomorphic sectional curvature k. Suppose the following conditions are satisfied:*

(i') *The first eigenvalue* λ_1 *of the Laplacian* Δ_g *acting on* $C^\infty(M)$ *satisfies the inequality:*

$$(n+1)k - (3n-1)f^2 - 2f' - \lambda_1 \geq 0;$$

(ii') $(n+1)(k+f^2) + 2(2n-1)f' \geq 0.$

Then the identity map 1_M *is unstable.*

Proof. Using Weitzenbock formula on $M \times \mathbb{R}$, we derive that the Laplacian Δ_1 of $(M \times \mathbb{R})$-valued 1-forms is given by(see [14, 29]):

$$\Delta_1 V = \bar{\Delta} V + \sum_{i=1}^{2n+1} R(V, E_i)E_i,$$

where $\{E_1, \ldots, E_n, E_{n+1} = \phi E_1, \ldots, E_{2n} = \phi E_n, E_{2n+1} = \xi\}$ is an adapted orthonormal frame on M and $\bar{\Delta}$ is the rough Laplacian. Therefore, in view of the above equation, we obtain from the second variation formula that the Hessian of the identity map $1_M : M \to M$ is

$$Hess_{1_M}(V, V) = \int_M g(\Delta_1 V, V)\vartheta_g - 2\sum_{i=1}^{2n+1} \int_M g(R(V, E_i)E_i, V)\vartheta_g.$$

Consequently, if λ_1 is the first eigenvalue of the Laplacian Δ_g and f is a nonconstant corresponding eigenfunction, due to the fact that $\Delta_1 df = d\Delta_g f = \lambda_1 df$, we derive easily that

$$\begin{aligned} Hess_{1_M}(V, V) &= \int_M \left[\lambda_1 + (n-1)k - (3n-1)f^2 - 2f'\right] g(V, V)\vartheta_g \\ &\quad - \int_M \left[(n+1)(k+f^2) + 2(2n-1)f'\right] \eta^2(V)\vartheta_g, \end{aligned}$$

for $V = \mathrm{grad} f$. From the above equation, we deduce immediately that $Hess_{1_M}(V, V) \leq 0$, provided that conditions (i') and (ii') are satisfied. Hence, the identity map 1_M is unstable if (i') and (ii') are both valid. □

Remark 4.3. It is easy to see that (28), (29) and (30) imply that the Ricci tensor Ric on M satisfies

$$\begin{aligned} Ric(X, X) &= \left[f' - \frac{(n+1)k - (3n-1)f^2}{2}\right] g(X, X) \\ &\quad + \left[\frac{(n+1)(k+f^2)}{2} + (2n-1)f'\right] \eta^2(X), \end{aligned} \tag{32}$$

for any vector field X on M. Replacing X by $X + Y$ in (32) and using the symmetry of Ric, we derive immediately that the Ricci tensor can be expressed as

$$\begin{aligned} Ric(X,Y) &= \left[f' - \frac{(n+1)k-(3n-1)f^2}{2}\right] g(X,Y) \\ &+ \left[\frac{(n+1)(k+f^2)}{2} + (2n-1)f'\right] \eta(X)\eta(Y), \end{aligned}$$

for any vector fields X, Y on M. Therefore, we deduce that the Ricci tensor on M satisfies

$$Ric = \alpha g + \beta \eta \otimes \eta, \tag{33}$$

where α and β are scalar functions given by

$$\alpha = f' - \frac{(n+1)k-(3n-1)f^2}{2} \tag{34}$$

and

$$\beta = \frac{(n+1)(k+f^2)}{2} + (2n-1)f'. \tag{35}$$

But this means that $M(\phi, \xi, \eta, g)$ is an η-Einstein manifold. Consequently, we have that any normal locally conformal cosymplectic manifold of dimension $2n+1 \geq 5$ and of pointwise constant ϕ-holomorphic sectional curvature k is an η-Einstein manifold with scalar functions α and β given by (34) and (35) (see also [27]). As an immediate consequence of this remark, we can see that the hypotheses of Theorems 4.1 and 4.2, which at first glance seemed very technical, are related to these scalar functions α and β. Moreover, having in mind these relations, we can rewrite Theorems 4.1 and 4.2 in a much simpler form, as follows.

Theorem 4.4. *Let $M(\phi, \xi, \eta, g)$ be a compact normal locally conformal cosymplectic manifold of dimension ≥ 5 and of pointwise constant ϕ-holomorphic sectional curvature. If the scalar functions α and β are both nonpositive, then the identity map 1_M is weakly stable.*

Theorem 4.5. *Let $M(\phi, \xi, \eta, g)$ be a compact normal locally conformal cosymplectic manifold of dimension ≥ 5 and of pointwise constant ϕ-holomorphic sectional curvature. If the scalar functions α and β are such that α is bounded above by $\frac{\lambda_1}{2}$ (where λ_1 is the first eigenvalue of the Laplacian Δ_g acting on $C^\infty(M)$) and β is nonnegative, then the identity map 1_M is unstable.*

Acknowledgements

This work was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS/CCCDI - UEFISCDI, project number PN-III-P4-ID-PCE-2020-0025, within PNCDI III.

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